Hydrodynamics of Wave Flow

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In this paper a method for obtaining periodical solutions of the nonlinear equations of motion is proposed. The method is applied to the case of a falling liquid film in wave motion. It consists in a triple series expansion; the first is a Taylor expansion, with respect to the distance y_1 from the free surface; a second one, representing the periodicity condition, is a Fourier expansion with respect to the variable $z \equiv k(x-ct)$, and a third one is a Taylor expansion with respect to the amplitude $2i\phi_1$. The calculation of the different coefficients is made easy by the fact that the algebraic equations obtained are linear and do not simultaneously contain all the unknowns. This allows the performance of the computation step by step in increasing order of the powers of ϕ_1. The periodicity condition allows the determination of all physical quantities as functions of one of them. The amplitude $|2i\phi_1|$ was selected as the parameter.

The existence of a dimensionless quantity for the wave flow is outlined. Arguments are adduced in support of the fact that the amplitude $|2i\phi_1|$ depends only on ψ and a universal curve $|2i\phi_1|$ vs. ψ is plotted on the basis of experimental data. Theoretical equations for the wave length, the wave velocity and the film thickness as a function of ψ are established. There is good agreement between the theoretical equations and experiment.

If a liquid film flows down a vertical plate it would be expected that the motion be laminar. Experiment shows, however, that the free surface of the film is not plane (Figure 1), but is disturbed by an unsteady periodical motion (*1* to *7*). The occurrence of this wave motion is due to the fact that the laminar steady motion is not stable to small perturbations and for this reason a new type of motion, stable to perturbations, is organized. Yih (*8*), Benjamin (*9*), Hanratty and Herschman (*10*), and Whitaker (*11*) have shown, by means of the linearized theory of instability, that the Nusselt velocity distribution is not stable to small perturbations; this theory however, is not able to provide information concerning the behavior of the stable state.

Such information has been obtained by Kapitza (*12, 13*) who postulated that the film thickness and the velocity components are periodical functions of the variable $z = k(x - ct)$ and looked for solutions of this type for a very approximate equation of motion.

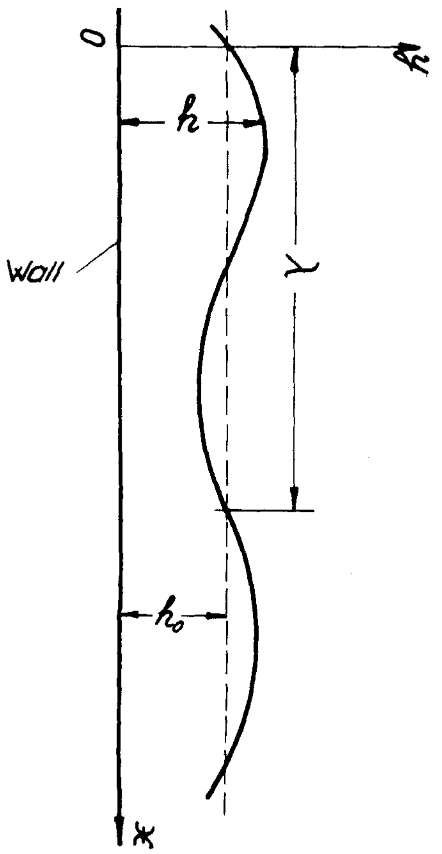

Fig. 1. Wave motion on a vertical plate.

The equations obtained in this manner by Kapitza for the film thickness, for the wave velocity, and for the wave length, lead to values of the same order of magnitude as the experimental ones.

However, Kapitza's theory predicts a dependence of the wave length on velocity, while experiments (*14* to *17*) show it to be practically independent of velocity; the theory also predicts a constant value for the ratio between the wave velocity and the average velocity of the liquid, while experiment shows that this ratio decreases with the increase of the liquid's average velocity.

It has been shown previously (*18*) that the use of the theoretical expressions given by Kapitza for the velocity distribution leads to a result, for the rate of mass transfer, which does not agree with experiment. While using Kapitza's results one obtains an increase of the mass transfer coefficient by 30%, yet the experiment shows a much larger increase of up to about 150% (both increases compare with the steady laminar case (*19* to *21*).

The disagreement between theory and experiment may be due to the assumption that the velocity components and the film thickness are periodical functions of z and/or to the simplifying assumptions made in the calculation. In the first assumption it may be noted that experiments (*15, 16*) show that the wave motion is a very complex one. Nevertheless, there exists a region in the vicinity of the point of wave inception (the position of this point depends on Reynolds number) of comparatively regular wave motion which may be used (and has been used) for determining experimentally wave lengths that have the usual physical meaning. The form of solution proposed by Kapitza is adequate, at least for this region; it assumes that the wave length, the wave velocity, and the amplitude are constant. For other regions and especially for larger Reynolds numbers more complex solutions, as are for instance those in which the above parameters depend on x, or those composed of clusters of interacting waves, might describe the hydrodynamic process in a more adequate manner. Therefore at least for the region mentioned above, the disagreement between Kapitza's theory and experiment is due to the simplifying assumptions made in the calculation. The main approximation is the assumption that the vertical velocity component u depends parabolically on the distance y from the wall, as in the Nusselt equation

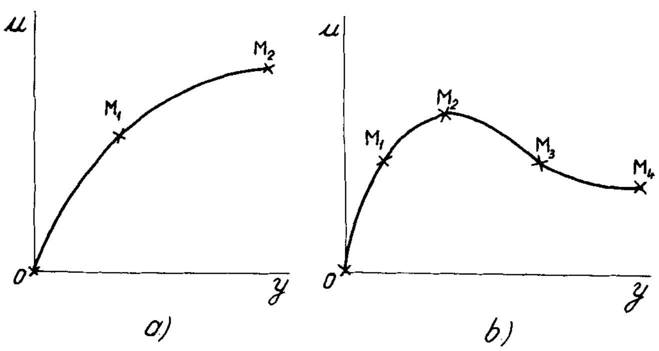

Fig. 2. Qualitative velocity distribution in a tube. (a) laminar flow, (b) unsteady periodical flow.

$$u = \frac{3\overline{u}_o(x,t)}{h(x,t)}\left(y - \frac{y^2}{2h(x,t)}\right)$$

That such an approximation is not entirely satisfactory results even from the second step in Kapitza's calculation. The insertion of the above expression in the equations of motion must be followed by averaging with respect to y, in order to obtain an equation for $\overline{u}_o(x,t)$. That the parabolic assumption is not a satisfactory one results also in the comparison with a somewhat similar case (*22* to *25*), namely that of a fluid flowing in a tube under the action of a pressure drop which depends periodically on time. Though in the case of a steady laminar motion the velocity distribution is parabolic, in the above mentioned case the velocity has a maximum in the vicinity of the wall. In order to take into account such an effect it is necessary to use a curve which should pass through at least five points (see Figure 2), that is, a curve at least of the fourth degree with respect to y.* In the case of wave flow the pressure has also a periodical time dependence and the above mentioned remark suggests the use of a polynomial in y at least of the fourth degree.

We shall propose in the following a method for obtaining periodical solutions of the nonlinear equations of motion that can be made as precise as one likes. The method shows that an expansion up to at least a fourth degree with respect to y is needed for the velocities. By using this method, an agreement with the experiment is obtained for the wave length and for the wave velocity.

Besides its intrinsic interest, the treatment of the hydrodynamics of wave motion of a thin liquid film may be useful in two other respects: (a) It appears to be the simplest case in which the stable state of motion is an unsteady one and a method of calculation, based on the periodicity condition and developed for its examination, may be useful for the solution of more complex unsteady cases (perhaps even for turbulent motions†). (b) By its means it is possible to treat the problem of mass transfer in wave motion. As mentioned above, Kapitza's velocity distribution for mass transfer leads to disagreement with the experiment; however, it was shown (*18*) that if one uses the experimental wave length, wave velocity, and amplitude, instead of the theoretical ones, the agreement may be much improved. The agreement with the experiment obtained in the present paper for the hydrodynamical parameters suggests that it is possible to obtain theoretical equations for mass transfer which represent the facts in a satisfactory manner. The problem was examined (*27*) and we notice here the good agreement with the experiment that was obtained.

* One can obtain such a curve with a cubic, although the maximum is then constrained to lie within certain values of y. With a quartic this constraint disappears.

† Approximate solutions in this direction have been suggested (*26*). They consist of a synthesis between renewal models and some results of the periodicity condition.

METHOD OF CALCULATION

From a mathematical point of view the problem lies in looking for periodical solutions of the system of equations which consist of the Navier-Stokes equations and the equation of continuity. By denoting u and v as the velocity components (see Figure 1) one may write

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (1b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1c)$$

The following boundary conditions must be satisfied:

$$u = 0, \quad v = 0 \quad \text{for} \quad y = 0 \quad (1d)$$

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{for} \quad y = h \quad (1e)$$

$$p_{xy}\cos 2\delta - \frac{1}{2}(p_{xx} - p_{yy})\sin 2\delta = 0 \quad \text{for} \quad y = h \quad (1f)$$

$$p_{xx}\sin^2\delta + p_{yy}\cos^2\delta - p_{xy}\sin 2\delta + p_a + p_\sigma = 0 \quad \text{for} \quad y = h \quad (1g)$$

where

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \quad p_{yy} = -p + 2\mu\frac{\partial v}{\partial y},$$

$$p_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \; p_\sigma = -\frac{\sigma}{(1+tg^2\delta)^{3/2}}\frac{\partial^2 h}{\partial x^2},$$

$$tg\delta = \frac{\partial h}{\partial x} \quad (1h)$$

The Equation (1*d*) expresses the condition of no slipping and of no net interfacial mass transfer at the solid-liquid interface, and the Equation (1*e*) expresses the condition that the free surface is streamline. Equations (1*f*) and (1*g*) express the equality of the shear stresses and the equality of the normal stresses at the free surface.

By taking into account the continuity equation, the boundary conditions (1*f*) and (1*g*) become

$$p - p_a = p_\sigma - \frac{2\mu}{\cos 2\delta}\frac{\partial u}{\partial x} \quad (1i)$$

$$\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial x}tg\,2\delta - \frac{\partial v}{\partial x} \quad (1j)$$

For sufficiently small film thickness, one may neglect (see Appendix IV) in Equation (1*b*) all terms containing v. There results $\partial p/\partial y = 0$ and therefore

$$p = p(x,t)$$

The pressure p is therefore given by its value at the boundary; Equation (1*i*) leads to

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\sigma}{\rho}\frac{\partial}{\partial x}\left\{\frac{1}{(1+tg^2\delta)^{3/2}}\cdot\frac{\partial^2 h}{\partial x^2}\right\}$$

$$-2\nu\frac{\partial}{\partial x}\left(\frac{1}{\cos 2\delta}\frac{\partial u}{\partial x}\right) \approx -\frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3} - 2\nu\frac{\partial^2 u}{\partial x^2}, \quad (y = h) \quad (2)$$

By the replacement of $\partial p/\partial x$ in Equation (1a) by its expression from Equation (2), and the neglect of the terms thus obtained which contain $\partial^2 u/\partial x^2$, as compared to $\partial^2 u/\partial y^2$, and by neglecting the right side in the boundary condition (1j), the system of Equation (1) becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3} + g + \nu\frac{\partial^2 u}{\partial y^2} \quad (3a)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3b)$$

$$u = v = 0 \quad \text{for} \quad y = 0 \quad (3c)$$

$$\frac{\partial u}{\partial y} = 0, \quad v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}, \quad \text{for} \quad y = h \quad (3d)$$

As shown in Appendix IV all the above approximations are valid of $kh_0 \leqq 0.3$.

Since one assumes that the liquid film has a wave motion which is propagating in the x direction, the velocities, u and v, and the thickness, h, of the film do not depend separately on x and t, but only via the combination

$$z \equiv k(x - ct) \quad (4)$$

For this reason, we shall choose z and y as independent variables. It is also convenient, as will result from the following, to use the change of variable

$$y_1 = h - y \quad (5)$$

Equations (3a) and (3b) become

$$-k(c-u)\frac{\partial u}{\partial z} + (kuh' - kch' - v)\frac{\partial u}{\partial y_1} = \frac{\sigma k^3}{\rho}h''' + g + \nu\frac{\partial^2 u}{\partial y_1^2} \quad (6a)$$

$$k\left(\frac{\partial u}{\partial z} + h'\frac{\partial u}{\partial y_1}\right) - \frac{\partial v}{\partial y_1} = 0 \quad (6b)$$

It is convenient to introduce the quantity

$$v_1 \equiv kuh' - kch' - v = \frac{dy_1}{dt} \quad (7)$$

which represents the horizontal velocity with respect to the free surface. The system (6) and the boundary conditions (3c) and (3d) may be written as

$$-k(c-u)\frac{\partial u}{\partial z} + v_1\frac{\partial u}{\partial y_1} = \frac{\sigma k^3}{\rho}h''' + g + \nu\frac{\partial^2 u}{\partial y_1^2} \quad (8a)$$

$$k\frac{\partial u}{\partial z} + \frac{\partial v_1}{\partial y_1} = 0 \quad (8b)$$

$$u = 0, \quad v_1 = -kch' \quad \text{for} \quad y_1 = h \quad (8c)$$

$$\frac{\partial u}{\partial y_1} = 0, \quad v_1 = 0 \quad \text{for} \quad y_1 = 0 \quad (8d)$$

The solutions of system (8) have to be periodical with respect to the variable z.

By denoting Q the average flow rate of the liquid and h_0 the average thickness of the film ($h_0 = \overline{h}$) we shall expand the velocities u and v_1 with respect to the variable y_1/h_0:

$$u = \frac{3}{2}\frac{Q}{h_0}\sum_{n=0}^{\infty} a_n\left(\frac{y_1}{h_0}\right)^n, \quad v_1 = \frac{3}{2}\frac{Q}{h_0}\sum_{n=0}^{\infty} b_n\left(\frac{y_1}{h_0}\right)^n \quad (9)$$

where the dimensionless coefficient a_n and b_n are periodical functions of z. It is also convenient to introduce the dimensionless quantities χ and ϕ defined by

$$c \equiv \chi\frac{Q}{h_0}, \quad h \equiv h_0(1 + \phi) \quad (10)$$

By introducing the expansion (9) in Equations (8) and identifying the terms in $(y_1/h_0)^n$, one obtains the following system which has an infinite number of equations:

$$-\chi a_n' + \frac{3}{2}\sum_{l=0}^{n} a_{n-l}\left(a_l' + \frac{n-l}{kh_0}b_{l+1}\right) = \frac{2}{3}\frac{h_0^2}{Q^2}\left(\frac{\sigma k^2 h_0}{\rho}\phi''' + \frac{g}{k}\right)\delta_{0,n}$$

$$+ \frac{(n+1)(n+2)\nu}{Qkh_0}a_{n+2}, \quad kh_0 a_n' + (n+1)b_{n+1} = 0,$$

$$\sum_{n=0}^{\infty} a_n(1+\phi)^n = 0, \quad \sum_{n=0}^{\infty} b_n(1+\phi)^n = -\frac{2}{3}\chi kh_0\phi'$$

$$a_1 = b_0 = 0, \quad n = 0, 1, 2, 3 \ldots \quad (11)$$

where $\delta_{0,n}$ is the Kronecker symbol

$$\delta_{0,n} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

By complete induction one may prove that (see Appendix I) from system (11) there results

— for $n = 2m + 1$ (odd)

$$a_{2m+1} = b_{2m} = 0, \quad m = 0, 1, 2 \ldots \quad (12a)$$

— for $n = 2m$ (even)

$$\sum_{m=0}^{\infty} a_{2m}(1+\phi)^{2m} = 0,$$

$$\sum_{m=0}^{\infty}\frac{a_{2m}(1+\phi)^{2m+1}}{2m+1} = \frac{2}{3}(1 + \chi\phi),$$

$$-\chi a'_{2m} + \frac{3}{2}\sum_{l=0}^{m}\frac{4l - 2m + 1}{2l+1}a'_{2l}\,a_{2m-2l}$$

$$= \frac{2}{3}\frac{h_0^2}{Q^2}\left(\frac{\sigma k^2 h_0}{\rho}\phi''' + \frac{g}{k}\right)\delta_{0,m} +$$

$$+ \frac{(2m+1)(2m+2)\nu}{Qkh_0}a_{2m+2}, \quad b_{2m+1} = -\frac{kh_0\,a'_{2m}}{2m+1},$$

$$m = 0, 1, 2 \ldots \quad (12b)$$

The system formed of Equations (12a) and (12b) together with the condition of periodicity of the solutions solve the problem of the wave flow; the required degree of accuracy is obtained by taking a sufficiently large number of equations from the system.

For simplification we shall restrict our calculation to the term of sixth degree with respect to y_1/h_0. In Appendix IV it is shown that such an approximation is valid if $\psi \leqq 5.4$.

In this case we must solve the system of equations:

$$\sum_{m=0}^{3} a_{2m}(1+\phi)^{2m} = 0,$$

$$\sum_{m=0}^{3} \frac{a_{2m}(1+\phi)^{2m+1}}{2m+1} = \frac{2}{3}(1+\chi\phi)$$

$$\left(\chi - \frac{3}{2}a_0\right)a'_0 + \frac{2h_0^2}{3Q^2}\left(\frac{\sigma k^2 h_0}{\rho}\phi''' + \frac{g}{k}\right) + \frac{2\nu a_2}{Qkh_0} = 0 \quad (13a)$$

$$\left(\chi - \frac{3}{2}a_0\right)a'_2 + \frac{3}{2}a'_0 a_2 + \frac{12\nu a_4}{Qkh_0} = 0$$

$$\left(\chi - \frac{3}{2}a_0\right)a'_4 + \frac{9}{2}a'_0 a_4 - \frac{1}{2}a_2 a'_2 + \frac{30\nu a_6}{Qkh_0} = 0$$

To these equations one has to add the equation of definition of the average thickness ($h_0 = \overline{h}$) which together with Equation (10) leads to

$$\overline{\phi} = 0 \quad (13b)$$

The calculation which follows is carried out by assuming that $|\phi| < 1$ (that is, the film does not break off).

For the quantities a_0, a_2, a_4, a_6 and ϕ we shall look for periodical solutions of z in the form of Fourier series:

$$a_0 = 1 + \sum_{-\infty}^{\infty} A_q e^{iqz},$$

$$a_2 = -1 + \sum_{-\infty}^{\infty} B_q e^{iqz},$$

$$a_4 = \sum_{-\infty}^{\infty} C_q e^{iqz},$$

$$a_6 = \sum_{-\infty}^{\infty} D_q e^{iqz},$$

$$\phi = \sum_{-\infty}^{\infty} \phi_q e^{iqz},$$

$$(i = \sqrt{-1}, \quad q = 0, \pm1, \pm2, \ldots \quad (14)$$

The first term in the expansions of a_0 and a_2 represents their values for the limiting case of a laminar motion.

Since the series (14) have complex coefficients, and a_0, a_2, a_4, a_6 and ϕ are real quantities, it is necessary that

$$A_{-q} = A_q^*, \quad B_{-q} = B_q^*, \quad C_{-q} = C_q^*, \quad D_{-q} = D_q^*, \quad \phi_{-q} = \phi_q^* \quad (15)$$

where we denoted by (*) the complex conjugate quantity. Therefore it is sufficient to perform the calculations only for positive values of q.

By introducing the expansion (14) in the system of Equations (13) and by denoting

$$\epsilon \equiv \frac{\nu}{Qkh_0}, \quad K \equiv \frac{\sigma k^2 h_0^3}{3\rho Q^2} \quad (16a)$$

one obtains

$$A_q + B_q + C_q + D_q - 2\phi_q = \alpha_q$$

$$A_q + \frac{1}{3}B_q + \frac{1}{5}C_q + \frac{1}{7}D_q - \frac{2}{3}\chi\phi_q = \beta_q$$

$$iq\left(\chi - \frac{3}{2}\right)A_q + 2\epsilon B_q - 2iq^3 K\phi_q + 2\delta_{0,q}\,\epsilon\left(\frac{gh_0^3}{3Q} - 1\right) = \gamma_q$$

$$-\frac{3i}{2}qA_q + iq\left(\chi - \frac{3}{2}\right)B_q + 12\epsilon C_q = \Gamma_q$$

$$\frac{iq}{2}B_q + iq\left(\chi - \frac{3}{2}\right)C_q + 30\epsilon D_q = \Delta_q$$

$$\phi_0 = 0, \quad (q = 0, \pm1, \pm2, \pm \ldots) \quad (16b)$$

where

$$\alpha_q = -\left[2\phi(a_2+1) + \phi^2 a_2 + a_4\sum_{p=1}^{4}\mathcal{E}_4^p\phi^p + a_6\sum_{p=1}^{6}\mathcal{E}_6^p\phi^p\right]_q$$

$$\beta_q = -\left[\phi(a_0+a_2) + a_2\left(\phi^2 + \frac{1}{3}\phi^3\right) + \frac{a_4}{5}\sum_{p=1}^{5}\mathcal{E}_5^p\phi^p + \frac{a_6}{7}\sum_{p=1}^{7}\mathcal{E}_7^p\phi^p\right]_q$$

$$\gamma_q = \frac{3}{2}[(a_0-1)a'_0]_q,$$

$$\Gamma_q = \frac{3}{2}[(a_0-1)a'_2 - (a_2+1)a'_0]_q$$

$$\Delta_q = \frac{3}{2}\left[(a_0-1)a'_4 - 3a'_0 a_4 + \frac{1}{3}(a_2+1)a'_2\right]_q \quad (16c)$$

$[\ ]_q$ represents the coefficient of e^{iqz} of the terms contained under the brackets and $\mathcal{E}_n^p = \dfrac{n!}{p!(n-p)!}$.

We remark that the left sides of Equations (16b) are linear with respect to A_q, B_q, $\ldots$ ϕ_q, while the right sides contain nonlinear terms of the form $A_p B_{q-p}$, $\phi_p \phi_r A_{q-p-r}$, $\ldots$

Restricting the expansions (14) to a certain finite value $q - 1$ the system (16b) leads to $(5q + 1)$ equations containing $(5q + 3)$ unknown quantities, A_q, B_q, C_q, D_q, ϕ_q, χ, h_0, k. Two quantities remain therefore undetermined.

Since the periodical quantities are determined only up to a phase constant one may choose, for this arbitrary phase constant, such a value that the calculation be simplified. For this reason we shall consider that in the expansion of ϕ, the coefficient of cos z is nil, thus

$$\phi_1 + \phi_{-1} = 0 \quad (17)$$

In this way only one quantity remains undetermined and we can express the other quantities as a function of it. We shall choose ϕ_1 as such a parameter, since it has a simple physical meaning in proportion with the first approximation of the wave amplitude. Indeed, in the first approximation we obtain

$$\phi = \phi_1 e^{iz} + \phi_{-1}e^{-iz}$$

which by taking into account Equation (17) becomes

$$\phi = 2i\phi_1 \sin z \quad (18)$$

We remark that since ϕ is a real quantity, the real part of ϕ_1 must be nil.

We shall use for the other quantities, expansions in a series of powers of ϕ_1. As shown in Appendix II, for the real quantities we have expansions of the form:

$$h_0 = \sum_{r=0}^{\infty} H_{2r}\,\phi_1^{2r}, \quad \chi = \sum_{r=0}^{\infty} \chi_{2r}\,\phi_1^{2r},\ k = \sum_{r=0}^{\infty} k_{2r}\,\phi_1^{2r}, \ldots \tag{19a}$$

in which there appear only the even powers of ϕ_1. (H_{2r}, χ_{2r}, k_{2r} are real coefficients.)

For the quantities A_q, B_q, $\ldots$, ϕ_q expansions of the form (see Appendix II)

$$A_q = \phi_1^{|q|} \sum_{r=0}^{\infty} A_{q,2r}\,\phi_1^{2r}, \quad B_q = \phi_1^{|q|} \sum_{r=0}^{\infty} B_{q,2r}\,\phi_1^{2r}, \ldots \tag{19b}$$

$$\phi_q = \phi_1^{|q|} \sum_{r=0}^{\infty} \phi_{q,2r}\,\phi_1^{2r}$$

are valid.

Concerning the expansions (19*b*) we notice that the coefficients A_q, B_q, $\ldots$ ϕ_q have the order of magnitude of $\phi_1^{|q|}$.

From the conditions of reality (15), one obtains

$$A_{-q,2r} = (-1)^{|q|} A^*_{q,2r}, \ldots, \tag{20}$$

$$\phi_{-q,2r} = (-1)^{|q|} \phi^*_{q,2r}, \ldots,$$

The coefficients $A_{q,2r}$, $\ldots$ $\phi_{q,2r}$, H_{2r}, $\ldots$ may be calculated by introducing the expansions (19) in the system formed of Equations (16*b*) and (17*a*) and identifying the terms in $\phi_1^{|q|+2r}$. There results:

$$A_{q,2r} + B_{q,2r} + C_{q,2r} + D_{q,2r} - 2\phi_{q,2r} = \alpha_{q,2r}$$

$$A_{q,2r} + \frac{1}{3} B_{q,2r} + \frac{1}{5} C_{q,2r} + \frac{1}{7} D_{q,2r} - \frac{2}{3}\ \chi_{2l}\phi_{q,2r-2l} = \beta_{q,2r}$$

$$gH_{2l}H_{2m}H_{2r-2l-2m} = 3\nu Q(\delta_{0,r} - B_{0,2r}) \quad (q = 0)$$

$$iq\ \left(\chi_{2l}A_{q,2r-2l} - \frac{3}{2} A_{q2,r}\right) + 2\epsilon_{2l}B_{q,2r-2l} - 2iK_0 q^3 \frac{k_{2l}}{k_0}\ \frac{k_{2m}}{k_0} (\phi_{q,2r-2l-2m} - B_{0,2s}\,\phi_{q,2r-2l-2m-2s}) = \gamma_{q,2r} \quad (q \neq 0) \tag{21}$$

$$-\frac{3i}{2} qA_{q,2r} + iq\ \left(\chi_{2l}B_{q,2r-2l} - \frac{3}{2} B_{q,2r}\right) + 12\epsilon_{2l}C_{q,2r-2l} = \Gamma_{q,2r}$$

$$\frac{iq}{2} B_{q,2r} + iq\left(\chi_{2l}C_{q,2r-2l} - \frac{3}{2} C_{q,2r}\right) + 30\epsilon_{2l}D_{q,2r-2l} = \Delta_{q,2r}$$

$$\phi_{0,2r} = 0, \quad \phi_{1,2r} = -\phi_{-1,2r} = \delta_{0,r}$$

$$(q = 0, \pm 1, \pm 2 \ldots;\quad r = 0, 1, 2 \ldots)$$

Indices l, m and s are summation indices; for instance $\epsilon_{2l}B_{2r-2l} \equiv \sum_{l=0}^{r} \epsilon_{2l}B_{2r-2l}$ and $H_{2l}H_{2m}H_{2r-2l-2m} \equiv \sum_{l=0}^{r}\sum_{m=0}^{r-l} H_{2l}H_{2m}H_{2r-2l-2m}$. The quantities $\alpha_{q,2r}$, $\beta_{q,2r}$, $\gamma_{q,2r}$, $\Gamma_{q,2r}$ and $\Delta_{q,2r}$ are the coefficients of the terms of degree $|q| + 2r$ in the ϕ_1-expansion of the quantities from expression (16*c*).

The above system of equations may be solved easily, since the unknown quantities result, step by step, in increasing order of power of ϕ_1. We remark that for a certain approximation of the order, $|q| + 2r$, one obtains a system of linear algebraic equations, the nonlinear terms $\alpha_{q,2r}$, $\beta_{q,2r}$, contain only the coefficients $A_{p,2s}$, $B_{p,2s}$ $\ldots$, $\phi_{p,2s}$, and whose indices $(|p| + 2s) < |q| + 2r$. The nonlinear terms may therefore be calculated by means of the preceding approximations. In this manner the expressions (19) achieve the solution of the nonlinear Navier-Stokes equations by means of an infinite number of linear systems of algebraic equations. The convergence in the considered range is rapid so that only a few systems of equations suffice.

RESULTS OF THE CALCULATION AND COMPARISON WITH EXPERIMENT

The zero order approximation is obtained for $q = r = 0$. There results

$$A_{00} = B_{00} = C_{00} = D_{00} = \phi_{00} = 0 \tag{22}$$

$$gH_0^3 = 3\nu Q$$

For H_0 (the first term in the expansion of h_0 with respect to the amplitude) one obtains the equation valid for a laminar steady flow.

In the first order approximation ($q = 1$, $r = 0$) one obtains

$$A_{10} + B_{10} + C_{10} + D_{10} = 2 \quad (\phi_{10} = 1)$$

$$A_{10} + \frac{1}{3} B_{10} + \frac{1}{5} C_{10} + \frac{1}{7} D_{10} = \frac{2}{3} \chi_0$$

$$\left(\chi_0 - \frac{3}{2}\right) A_{10} + 2i\epsilon_0 B_{10} = 2K_0 \tag{23}$$

$$-\frac{3}{2} A_{10} + \left(\chi_0 - \frac{3}{2}\right) B_{10} - 12i\epsilon_0 C_{10} = 0$$

$$\frac{1}{2} B_{10} + \left(\chi_0 - \frac{3}{2}\right) C_{10} - 30i\epsilon_0 D_{10} = 0$$

ϵ_0 and K_0 represent the first coefficients of the series expansion of the quantities, ϵ and K, [Equations (16*a*)] with respect to ϕ_1.

By solving the system formed of the first four equations of system (23), and introducing the expressions thus obtained for A_{10}, B_{10}, C_{10} and D_{10} in the fifth equation, one obtains, owing to the complex quantities involved, two equations:

$$K_0\left(\frac{6\chi_0^2}{7} - \frac{16\chi_0}{7} + \frac{51}{35}\right) = \left(\chi_0 - \frac{3}{2}\right)^3 \left(\frac{\chi_0}{3} - \frac{1}{7}\right) - 30\,\epsilon_0^2\left(4\chi_0^2 - 11\chi_0 + \frac{33}{5} - 8K_0\right)$$

$$\left(\chi_0 - \frac{3}{2}\right)\left(\frac{5\chi_0^2}{3} - \frac{10\chi_0}{3} + \frac{10}{7}\right) = 40\,\epsilon_0^2(\chi_0 - 3) + K_0\left(4\chi_0 - \frac{29}{7}\right) \tag{24}$$

From the system (24) one obtains χ_0 and K_0 as functions of ϵ_0; they are plotted in Figures 3 and 4. In these figures, there appear for comparison, curves computed by means of the fourth-order expansion with respect to y_1 (dashed curve). We remark that for large values of ϵ_0 (small values of Reynolds number) the two approximations lead to practically the same results. By means of

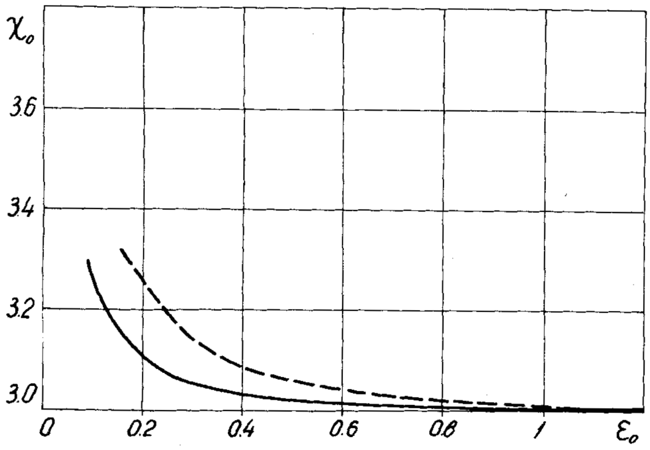

Fig. 3. χ_0 vs. ϵ_0.
——— sixth approximation with respect to y_1.
– – – – – fourth approximation with respect to y_1.

χ_0, H_0 and K_0 one may calculate k_0 and c_0 as

$$k_0 = \left(K_0 \frac{g\rho Q}{\nu\sigma} \right)^{1/2}, \quad c_0 = \frac{Q\chi_0}{H_0} \tag{25}$$

By using the general Equations (21) one may perform the calculation of the coefficients which enter in the second-order ($q = 0$, $2r = 2$; $q = 2$, $r = 0$), the third-order ($q = 1$, $2r = 2$; $q = 3$, $r = 0$), etc., approximations with respect to ϕ_1. We have carried out the calculation up to the third-order approximation inclusively (see Appendix III).

The first correction for the film thickness is obtained in the second approximation ($q = 0$, $2r = 2$). One obtains

$$\frac{H_2}{H_0} \approx 2$$

and therefore

$$\frac{h_0}{H_0} = 1 + 2\phi_1^2 \tag{26}$$

The first corrections for the wave number and for the wave velocity are obtained in the third approximation ($q = 1$, $2r = 2$). In Figure 5 we have plotted the ratios k_2/k_0 and χ_2/χ_0, which enter in the expansions of k and χ, as functions of ϵ_0. The wave number k and the wave velocity c may be calculated by aid of the following:

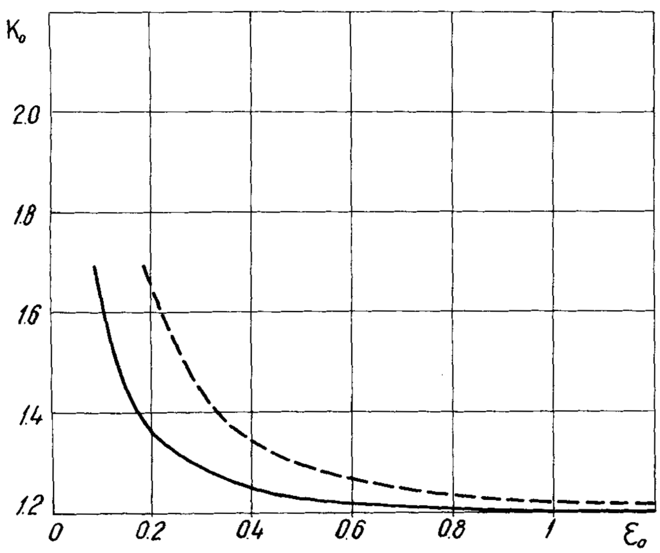

Fig. 4. K_0 vs. ϵ_0.
——— sixth approximation with respect to y_1.
– – – – – fourth approximation with respect to y_1.

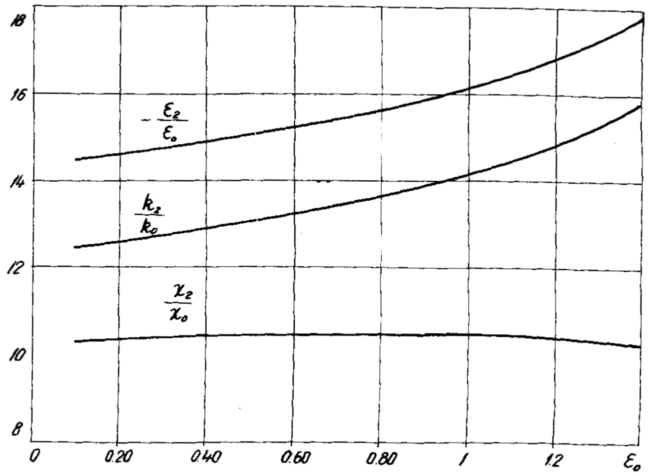

Fig. 5. $\frac{\epsilon_2}{\epsilon_0}$, $\frac{k_2}{k_0}$ and $\frac{\chi_2}{\chi_0}$ vs. ϵ_0.

$$\frac{k}{k_0} = 1 + \frac{k_2}{k_0}\phi_1^2, \tag{27}$$

$$\frac{c}{c_0} = 1 + \frac{c_2}{c_0}\phi_1^2 = 1 + \left(\frac{\chi_2}{\chi_0} - \frac{H_2}{H_0} \right) \phi_1^2 \tag{28}$$

The most important conclusion is that all dimensionless coefficients (H_{2r}/H_0, k_{2r}/k_0, $\ldots$, $A_{q,2r}$, $B_{q,2r}$, $\ldots$, $\phi_{q,2r}$) appearing in the expansions with respect to ϕ_1 are functions of a single dimensionless parameter

$$\epsilon_0 \equiv \frac{\nu}{Qk_0H_0}$$

By substituting H_0 and k_0 by their expressions (22) and (25) and taking into account that K_0 depends only on ϵ_0 (Figure 4) one obtains

$$\epsilon_0 = F\left(\left(\frac{\rho}{\sigma} \right)^{1/2} \frac{g^{1/6} Q^{11/6}}{\nu^{7/6}} \right) \equiv F(\psi) \tag{29}$$

The function F is plotted in Figure 6; it tends asymptotically towards the line $\epsilon_0 = 0.632\,\psi^{-1}$.

Therefore all the above mentioned dimensionless coefficients are functions only of the dimensionless parameter

$$\psi = \left(\frac{\rho}{\sigma} \right)^{1/2} \frac{g^{1/6} Q^{11/6}}{\nu^{7/6}} \tag{30}$$

In order to be able to obtain general curves for the wave number, wave velocity, film thickness, and velocity distribution it is necessary to know the values of the amplitude ϕ_1. For the determination of this quantity two ways seem

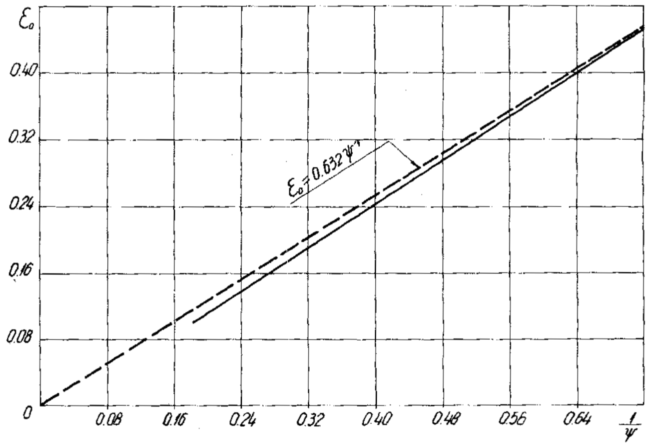

Fig. 6. ϵ_0 vs. ψ^{-1}.

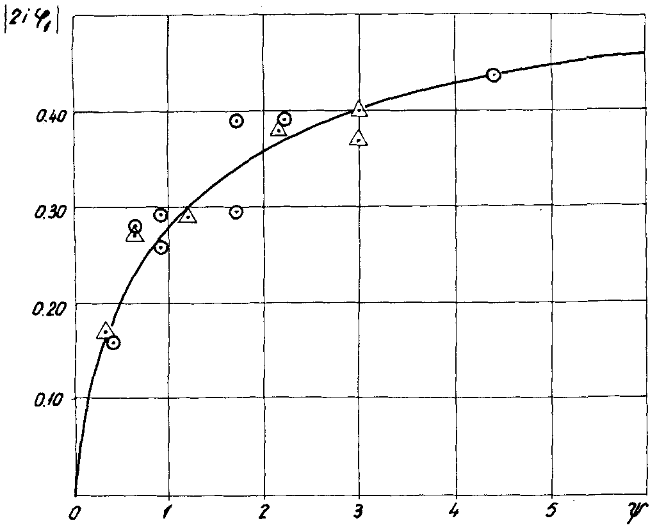

Fig. 7. $|2i\phi_1|$ **vs.** ψ. ○ = **experimental points for water** *(14)*. △ — **experimental point for ethyl alcohol** *(14)*.

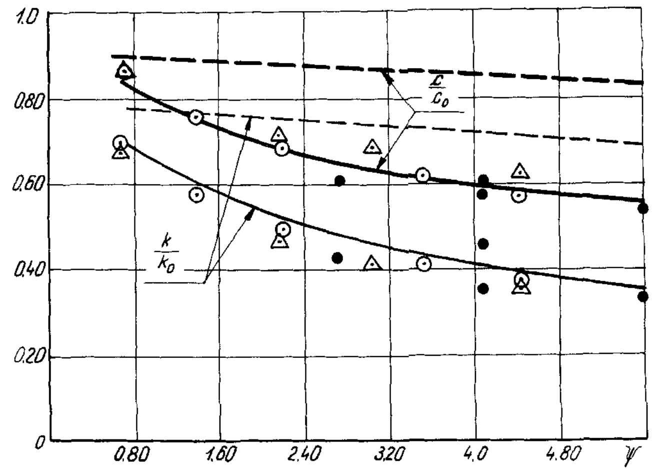

Fig. 8. $\frac{k}{k_0}$ **and** $\frac{c}{c_0}$ **vs.** ψ

——— **present theory**
– – – – – **Kapitza's theory**
⊙ ***Kapitza's experimental points for water at 14°C.***
● **Jones and Whitaker's experimental points for water at 25°C.**
△ **Kapitza's experimental points for ethyl alcohol.**

possible. One of them may be based on an arbitrary principle, outside hydrodynamics. Such a principle is, for instance, that of the minimum viscous dissipation of the mechanical energy used by Kapitza. The second way may be based on nonlinear stability considerations for the solutions of the equations of motion. We shall examine this problem in another paper. In regard to the first way, it may be remarked that an extremum condition may also be imposed to other physical quantities as are, for instance, h_0, k, the average value of the free surface, etc. The values obtained by us in this manner for ϕ_1 differ from one another. For this reason the problem of the determination of the amplitude is unsolved for the time being. Nevertheless, the above conclusion permits one to obtain some information concerning the amplitude. Indeed, whichever is the physical quantity chosen for the determination of ϕ_1 by means of an extremum condition, its average value may be written in the form

$$f = f_0(1 + f_2(\psi)\phi_1^2 + f_4(\psi)\phi_1^4 + \ldots) \qquad (31)$$

where f_0 represents the value of f for the laminar steady case, ($\phi_1 = 0$) and the dimensionless quantities f_2 and f_4 depend only on ψ. The extremum condition leads to

$$f_2(\psi) + 2f_4(\psi)\phi_1^2 + \ldots = 0 \qquad (32)$$

and consequently ϕ_1 is a function only of ψ. This remark suggests one to plot the experimental amplitude as a function of ψ. As it is shown in Figure 7, the experimental results obtained by Kapitza for water and ethyl alcohol are well fitted by a single curve in agreement with the above. For this reason we shall use this curve for the calculation of the other physical parameters.

Equations (27) and (28) and Figures 5 and 7 allow one to obtain the universal curves k/k_0 vs. ψ and c/c_0 vs. ψ. They are plotted in Figure 8 together with the experimental points obtained by Kapitza (*14*), and by Jones and Whitaker (*17*). It may be seen that the agreement is good. Tailby and Portalski's data (*15*) are not taken into account since they are outside the range of validity of the present theory.

In Table 1 a comparison is made between Kapitza's theory and the present theory and experiment. Experiment shows that the wave number is practically independent on the flow rate, Q, in agreement with our theory, while Kapitza's theory predicts an appreciable increase with the flow rate. In regard to the wave velocity, the present theory agrees better than that of Kapitza with the experimental results.

TABLE 1

Liquid	$Re = \frac{4Q}{\nu}$	χ	Kapitza's experiments k (cm.$^{-1}$)	Kapitza's experiments c (cm./sec.)	Jones-Whitaker's experiments k	Jones-Whitaker's experiments c	Kapitza's theory k	Kapitza's theory c	Present theory k	Present theory c
Water at 14°C.	28	0.64	7.40	14.70	—	—	9.20	14.70	7.50	14.50
	54	2.20	7.40	18.50	—	—	12.80	23.00	7.90	18.60
	80	4.40	7.40	21.00	—	—	15.40	30.00	7.90	20.90
Water at 25°C.	30	0.64	—	—	6.00	12.00	8.70	14.70	7.90	13.60
	66	2.70	—	—	7.20	17.40	12.80	24.80	8.50	18.80
	84	4.10	—	—	7.80	19.40	14.30	28.50	8.90	21.10
	98	5.60	—	—	7.70	20.80	15.60	32.30	7.90	21.30
Ethyl alcohol at 14°C.	17.50	0.64	8.90	12.80	—	—	11.50	12.90	9.80	12.60
	34	2.20	8.90	17.00	—	—	16.10	20.00	10.30	16.50
	50	4.40	8.90	20.00	—	—	19.40	26.00	10.00	18.40

We notice that the disagreement between the experimental rate of mass transfer and the theoretical one, obtained by means of Kapitza's hydrodynamic theory, is especially due to the ratio between the wave velocity and the average velocity (χ). While Kapitza obtains a constant value of 2.4 for this ratio, experiment shows a variation from 1.6 up to 2.2, in agreement with our results. For this reason it is to be expected that the present equations should lead to a better agreement with experiment for mass transfer. This was confirmed recently (27).

SUMMARY

The results obtained in this paper may be summarized as follows:

1. A method for obtaining periodical solutions of the equations of motion is proposed. It consists of replacing, by means of a triple series expansion, the nonlinear equations of motion by an infinite number of linear algebraic equations, which may be solved step by step. The first is a Taylor expansion with respect to y_1; the second, representing the periodicity condition, is a Fourier expansion with respect to the variable $z \equiv k(x - ct)$ and allows the determination of all the physical quantities as functions of one of them; and the third is a Taylor expansion with respect to the amplitude ϕ_1 which was selected as the undetermined quantity.

2. The most important result is that all dimensionless physical quantities (H_{2r}/H_0, k_{2r}/k_0, c_{2r}/c_0, ... $A_{q,2r}$, ... $\phi_{q,2r}$) are functions of a single dimensionless quantity ψ.

3. Theoretical equations for the wave length, wave velocity, and film thickness as functions of ψ and ϕ_1 are established up to the third-order approximation with respect to ϕ_1.

4. Arguments are adduced in support of the fact that ϕ_1 depends also only on ψ, and on the basis of experimental results a universal curve $|2i\phi_1|$ vs. ψ is given.

5. Universal curves k/k_0 and c/c_0 vs. ψ are plotted. They agree with experiment.

NOTATION

$a_n\, b_n$ = coefficients in the expansion (9)
A_q, B_q, C_q, D_q = dimensionless coefficients in the expansion (14)
$c = \omega/k$ = wave velocity; c_0, c_2, c_4 ... are the coefficients in the expansion of c with respect to ϕ_1
E = dimensionless coefficient defined by Equation (IV.11)
g = gravity acceleration
h = film thickness
h_0 = average film thickness; H_0, H_2, H_4 ... are the coefficients in the expansion of h_0 with respect to ϕ_1
i = $\sqrt{-1}$
k = wave number; k_0, k_2, k_4 ... are the coefficients in the expansion of k with respect to ϕ_1
K = $\sigma k^2 h_0^3/3\rho Q^2$
p = pressure
p_n = coefficients in the expansion (IV.1)
p_a = pressure above the free surface
Q = average liquid flow rate (cc./cm. sec.)
R = radius of curvature
t = time
u = vertical velocity component
$\overline{u}_0$ = average velocity with respect to y
v = horizontal velocity component
v_1 = component with respect to the free surface
x, y = rectangular coordinates (Figure 1)
y_1 = $h - y$
z = $k(x - ct)$

Greek Letters

α_q, β_q, γ_q, Γ_q and Δ_q = quantities defined by Equation (16b)
δ = arc $tg\ \partial h/\partial x$
$\epsilon \equiv \nu/Qkh_0$; ϵ_0, ϵ_2, ϵ_4 .. = are the coefficients in the expansion of ϵ with respect to ϕ_1
$\lambda = 2\pi/k$ = wave length
μ = dynamic viscosity
ν = kinematic viscosity
ρ = liquid density
σ = surface tension
ϕ = ratio between elongation and film thickness
ϕ_1 = coefficient of e^{iz} in the Fourier expansion of ϕ
$\chi = ch_0/Q$, χ_0, χ_2 = are the coefficients in the expansion of χ with respect to ϕ_1
ω = wave pulsation

Superscripts

$'$ = the first derivative with respect to z
$'''$ = the third derivative with respect to z
$\overline{}$ = temporal average
$*$ = complex conjugate quantity
$\wedge$ = real part of a complex quantity
$\sim$ = imaginary part of a complex quantity

Subscripts

n = index in the expansion with respect to y_1, Equation (9)
q = index in the Fourier expansion, Equation (14)
r = index in the expansion with respect to ϕ_1, Equation (19)

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Manuscript received June 30, 1967; revision received October 25, 1967; paper accepted October 26, 1967.

APPENDIX I

We shall show by complete induction that all terms a_n having an odd index ($n = 2m + 1$) are nil.

Indeed, let us assume

$$a_1 = a_3 = \ldots = a_{2m-1} = 0 \tag{I.1}$$

By eliminating b_{l+1} between the first two equations of the system (11) and by taking into account (I.1) one obtains

$$a_{2m+1} = 0 \tag{I.2}$$

Since $(\partial u/\partial y)_{y,h} = 0$, there results that $a_1 = 0$ and consequently all terms having an odd index are nil.

The system (11) together with Equation (I.2) lead to Equations (12). The second Equation (12b) is equivalent to

$$\sum_{n=0}^{\infty} b_n(1+\phi)^n = -\frac{2}{3}\chi k h_0 \phi'$$

Indeed, we may write successively:

$$\sum_{n=0}^{\infty} b_n(1+\phi)^n =$$

$$\sum_{n=0}^{\infty} [b_{2m}(1+\phi)^{2m} + b_{2m+1}(1+\phi)^{2m+1}] =$$

$$= -kh_0 \sum_{m=0}^{\infty} \frac{a'_{2m}(1+\phi)^{2m+1}}{2m+1} =$$

$$-kh_0 \sum_{m=0}^{\infty} \left[\frac{a'_{2m}(1+\phi)^{2m+1}}{2m+1} + \phi' a_{2m}(1+\phi)^{2m} \right] =$$

$$-kh_0 \sum_{m=0}^{\infty} \left[\frac{a_{2m}(1+\phi)^{2m+1}}{2m+1} \right]' = -\frac{2}{3}\chi h_0 k \phi' \tag{I.3}$$

therefore

$$\sum_{m=0}^{\infty} \frac{a_{2m}(1+\phi)^{2m+1}}{2m+1} = \frac{2}{3}\chi\phi + \text{const.} \tag{I.4}$$

where const. is an integration constant.

On the other hand we have

$$\overline{Q = \int_o^h u dy_1} = \frac{3}{2} Q \overline{\int_o^{(1+\phi)} \sum_{m=0}^{\infty} a_{2m} \left(\frac{y_1}{h_0}\right)^{2m} d\left(\frac{y_1}{h_0}\right)}$$

$$= \frac{3}{2} Q \sum_{m=0}^{\infty} \frac{\overline{a_{2m}(1+\phi)^{2m+1}}}{2m+1} \tag{I.5}$$

Therefore

$$\sum_{m=0}^{\infty} \frac{\overline{a_{2m}(1+\phi)^{2m+1}}}{2m+1} = \frac{2}{3} \tag{I.6}$$

Since $\overline{\phi} = 0$, from Equation (I.4) and (I.6) there results

$$\text{const.} = \frac{2}{3}$$

and therefore

$$\sum_{m=0}^{\infty} \frac{a_{2m}(1+\phi)^{2m+1}}{2m+1} = \frac{2}{3}(1+\chi\phi) \tag{I.7}$$

APPENDIX II

Let us expand ϕ_q, A_q, B_q, C_q and D_q in Taylor series with respect to ϕ_1, in the form

$$\phi_q = \sum_{r=0}^{\infty} \phi_{q,r}\,\phi_1{}^r$$

$$A_q = \sum_{r=0}^{\infty} A_{q,r}\,\phi_1{}^r, \ldots; \quad D_q = \sum_{r=0}^{\infty} D_{q,r}\,\phi_1{}^r \tag{II.1}$$

Since in the laminar case ($\phi_1 = 0$) we have $\phi = 0$, there results

$$\phi_{q,0} = 0 \tag{II.2}$$

From the system (16b) and taking into account (II.2), one obtains

$$A_{00} = B_{00} = C_{00} = D_{00} = 0 \tag{II.3}$$

It can be proven by complete induction that other coefficients in the expansion (II.1) are also nil, namely the coefficients

$$A_{q,r} = B_{q,r} = \ldots = \phi_{q,r} = 0 \quad \text{for} \quad r < |q|, \tag{II.4a}$$

$$A_{q,r} = B_{q,r} = \ldots = \phi_{q,r} = 0 \quad \text{for} \quad r = |q| + 2s + 1, \tag{II.4b}$$

$$h_{0,r} = \chi_r = \epsilon_r = k_r = c_r = 0 \quad \text{for} \quad r = 2s + 1 \tag{II.4c}$$

$$(s = 0, 1, 2 \ldots; \quad q = 0, \pm 1, \pm 2 \ldots)$$

Let us assume that Equations (II.4) are valid up to the index $r = |q| + 2s + 1$. By identifying in the system (16b) the terms that contain $\phi_1{}^{r+2} = \phi_1{}^{|q|+2s+3}$ and by taking into account that $\phi_0 = 0$, one obtains

$$A_{q,r+2} + B_{q,r+2} + C_{q,r+2} + D_{q,r+2} - 2\phi_{q,r+2} = 0$$

$$A_{q,r+2} + \frac{1}{3}B_{q,r+2} + \frac{1}{5}C_{q,r+2} + \frac{1}{7}D_{q,r+2} - \frac{2\chi_0}{3}\phi_{q,r+2} = 0$$

$$iq\left(\chi_0 - \frac{3}{2}\right)A_{q,r+2} + 2\epsilon_0 B_{q,r+2} - 2iq^3 K_0(1 - B_{00})\phi_{q,r+2} = 0 \quad (q \neq 0)$$

$$-\frac{3i}{2}qA_{q,r+2} + iq\left(\chi_0 - \frac{3}{2}\right)B_{q,r+2} + 12\epsilon_0 C_{q,r+2} = 0$$

$$\frac{iq}{2}B_{q,r+2} + iq\left(\chi_0 - \frac{3}{2}\right)C_{q,r+2} + 30\epsilon_0 D_{q,r+2} = 0$$

$$(r = |q| + 2s + 1, \quad s = 0, 1, 2, \ldots) \tag{II.5}$$

In the case $q \neq 0$, the system (II.5) is a homogeneous linear system of five equations containing five unknowns; in the case $q = 0$ the third equation in the system (II.5) is no longer valid, however from the condition $\phi_0 = 0$ there results $\phi_{0,r+2} = 0$, so that for $q = 0$ the system (II.5) reduces to a system having four equations containing four unknowns. In both cases it may be verified that for $\epsilon_0 \neq 0$ the determinant of the system differs from zero. Therefore

$$A_{q,r+2} = B_{q,r+2} = \ldots = \phi_{q,r+2} = 0 \tag{II.6}$$

Since for $q = 0$ and $r = 1$ one obtains by direct calculation

$$A_{01} = B_{01} = \ldots = \phi_{01} = 0$$

one may conclude that Equation (II.4b) is valid.

In a similar manner one may prove, by using the remaining Equations (16), that (II.4c) are valid too.

Consequently, the quantities A_q, B_q, $\ldots$, ϕ_q, h_0, χ and c may be expanded under the form (19).

APPENDIX III

For $q = 0$, $2r = 2$, from Equations (21) one obtains

$$\epsilon_0 D_{02} = \frac{2}{5}(\hat{A}_{10}\tilde{C}_{10} - \tilde{A}_{10}\hat{C}_{10}), \; \epsilon_0 C_{02} = \frac{1}{2}(\hat{A}_{10}\tilde{B}_{10} - \tilde{A}_{10}\hat{B}_{10}),$$

$$B_{02} = -\frac{3}{2}\left(2 + \frac{4}{5}C_{02} + \frac{6}{7}D_{02} - \alpha_{02}\right),$$

$$A_{02} = -(B_{02} + C_{02} + D_{02}) + \alpha_{02}$$

$$\frac{H_2}{H_0} = -\frac{1}{3}B_{02} \approx 2,$$

$$\alpha_{02} = 2(2\hat{B}_{10} + 4\hat{C}_{10} + 6\hat{D}_{10} - 1) \quad \text{(III.1)}$$

where $\hat{A}$ and $\tilde{A}$ are, respectively, the real and the imaginary parts of the complex quantity $A = \hat{A} + i\tilde{A}$.

For $q = 2$, $r = 0$, from Equations (21) there results

$$A_{20} + B_{20} + C_{20} + D_{20} - 2\phi_{20} = 1 - 2B_{10} - 4C_{10} - 6D_{10}$$

$$A_{20} + \frac{1}{3}B_{20} + \frac{1}{5}C_{20} + \frac{1}{7}D_{20} - \frac{2}{3}\chi_0\phi_{20} = -1$$

$$\beta_{12} = 2(1 + \phi_{20}) - 4(\hat{B}_{10} + 2\hat{C}_{10} + 3\hat{D}_{10}) + B^*_{10} + 2C^*_{10} + 3D^*_{10},$$

$$\gamma_{12} = \frac{3i}{2}(A_{02}A_{10} - A_{20}A^*_{10}),$$

$$\Gamma_{12} = \frac{3i}{2}(A_{02}B_{10} - B_{02}A_{10} - 3B_{20}A^*_{10} + 3A_{20}B^*_{10}),$$

$$\Delta_{12} = \frac{i}{2}(B_{02}B_{10} - B_{20}B^*_{10} + 3A_{02}C_{10} - 9C_{02}A_{10} - 15C_{20}A^*_{10} + 21A_{20}C^*_{10}) \quad \text{(III.4)}$$

The system (III.3) may be assimilated to a linear system of five equations having four unknowns: A_{12}, B_{12}, C_{12} and D_{12}. The compatibility condition leads to the following:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 12 \\ 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \left(\beta_{12} + \frac{2\chi_2}{3}\right) \\ i\left(\chi_0 - \frac{3}{2}\right) & 2\epsilon_0 & 0 & 0 & \left[\gamma_{12} + \frac{4iK_0}{3}B_{02} - 2\left(B_{10} + \frac{2iK_0}{\epsilon_0}\right)\epsilon_2 - iA_{10}\chi_2\right] \\ -\frac{3i}{2} & i\left(\chi_0 - \frac{3}{2}\right) & 12\epsilon_0 & 0 & (\Gamma_{12} - 12\epsilon_2 C_{10} - iB_{10}\chi_2) \\ 0 & \frac{i}{2} & i\left(\chi_0 - \frac{3}{2}\right) & 30\epsilon_0 & (\Delta_{12} - 30\epsilon_2 D_{10} - iC_{10}\chi_2) \end{vmatrix} = 0 \quad \text{(III.5)}$$

$$i(2\chi_0 - 3)A_{20} + 2\epsilon_0 B_{20} - 16iK_0\phi_{20} = \frac{3i}{2}A_{10}^2 \quad \text{(III.2)}$$

$$-3iA_{20} + i(2\chi_0 - 3)B_{20} + 12\epsilon_0 C_{20} = 0$$

$$iB_{20} + i(2\chi_0 - 3)C_{20} + 30\epsilon_0 D_{20} = \frac{i}{2}(B_{10}^2 - 6A_{10}C_{10})$$

In the calculation of the third order approximation two cases must be taken into account:

For $q = 1$, $2r = 2$, from system (21) we have

$$A_{12} + B_{12} + C_{12} + D_{12} = \alpha_{12}$$

$$A_{12} + \frac{1}{3}B_{12} + \frac{1}{5}C_{12} + \frac{1}{7}D_{12} - \frac{2\chi_2}{3} = \beta_{12}$$

$$i\left(\chi_0 - \frac{3}{2}\right)A_{12} + 2\epsilon_0 B_{12} + iA_{10}\chi_2 + 2\left(\frac{2iK_0}{\epsilon_0} + B_{10}\right)\epsilon_2 = \gamma_{12} + \frac{4iK_0}{3}B_{02}$$

$$-\frac{3i}{2}A_{12} + i\left(\chi_0 - \frac{3}{2}\right)B_{12} + 12\epsilon_0 C_{12} + iB_{10}\chi_2 + 12C_{10}\epsilon_2 = \Gamma_{12}$$

$$\frac{i}{2}B_{12} + i\left(\chi_0 - \frac{3}{2}\right)C_{12} + 30\epsilon_0 D_{12} + iC_{10}\chi_2 + 30D_{10}\epsilon_2 = \Delta_{12} \quad \text{(III.3)}$$

where

$$\alpha_{12} = -2(B_{02} + 2C_{02} + 3D_{02}) + 2(B^*_{10} + 2C^*_{10} + 3D^*_{10} - 1)\phi_{20} + 2(B_{20} + 2C_{20} + 3D_{20}) + 6C_{10} + 15D_{10} + B_{10} + 2(\hat{B}_{10} + 6\hat{C}_{10} + 15\hat{D}_{10}),$$

Equation (III.5) contains complex quantities and consequently it leads to two equations. From these equations one may calculate χ_2 and ϵ_2 as functions of ϵ_0 (Figure 5). Then the values of k_2 and c_2 are obtained by means of the following:

$$\frac{k_2}{k_0} = -\frac{\epsilon_2}{\epsilon_0} + \frac{1}{3}B_{02}, \quad \frac{c_2}{c_0} = \frac{\chi_2}{\chi_0} + \frac{1}{3}B_{02} = \frac{\chi_2}{\chi_0} - \frac{H_2}{H_0}$$

A_{12}, B_{12}, C_{12} and D_{12} result from the system (III.3).

For $q = 3$, $r = 0$ we have

$$A_{30} + B_{30} + C_{30} + D_{30} - 2\phi_{30} = \alpha_{30}$$

$$A_{30} + \frac{1}{3}B_{30} + \frac{1}{5}C_{30} + \frac{1}{7}D_{30} - \frac{2\chi_0}{3}\phi_{30} = \beta_{30}$$

$$3i\left(\chi_0 - \frac{3}{2}\right)A_{30} + 2\epsilon_0 B_{30} - 54iK_0\phi_{30} = \gamma_{30} \quad \text{(III.6)}$$

$$-\frac{9i}{2}A_{30} + 3i\left(\chi_0 - \frac{3}{2}\right)B_{30} + 12\epsilon_0 C_{30} = \Gamma_{30}$$

$$\frac{3i}{2}B_{30} + 3i\left(\chi_0 - \frac{3}{2}\right)C_{30} + 30\epsilon_0 D_{30} = \Delta_{30}$$

where

$$\alpha_{30} = -2(B_{10} + 2C_{10} + 3D_{10} - 1)\phi_{20} - 2(B_{20} + 2C_{20} + 3D_{20}) - B_{10} - 6C_{10} - 15D_{10}$$

$$\beta_{30} = -\frac{2}{3} - 2\phi_{20} + B_{10} + 2C_{10} + 3D_{10},$$

$$\gamma_{30} = \frac{9i}{2}A_{10}A_{20}, \quad \Gamma_{30} = \frac{3i}{2}(A_{10}B_{20} - A_{20}B_{10}),$$

$$\Delta_{30} = -\frac{3i}{2}(B_{10}B_{20} + A_{10}C_{20} + 5A_{20}C_{10}) \quad \text{(III.7)}$$

The system (III.6) gives the quantities A_{30}, B_{30}, C_{30}, D_{30}, and ϕ_{30}. In this manner all the quantities that appear up to the third-order approximation may be calculated.

APPENDIX IV

In order to establish the validity conditions of the approximations made in the calculations we shall use the exact system of equations and the exact boundary conditions. By introducing the expansion (9) for the velocity components and a similar expansion for the pressure

$$\frac{p - p_a}{\rho\left(\frac{3}{2}\frac{Q}{h_0}\right)^2} = \sum_{n=0}^{\infty} p_n \left(\frac{y_1}{h_0}\right)^n \tag{IV.1}$$

in the exact system of equation and by identifying the terms in $(y_1/h_0)^n$ there result an exact system having an infinite number of equations.

For $n = 0$ from the Navier-Stokes equations one obtains

$$-kh_0\left(\frac{2}{3}\chi - a_0\right)a_0 = -\left(\frac{\partial p_0}{\partial z} + p_1\phi'\right)kh_0 + \frac{2\nu}{3Q}\left[2a_2 + (kh_0)^2\left(a_0'' + 2a_1\phi'^2\right)\right] + \frac{4gh_0^3}{9Q^2} \tag{IV.2}$$

$$p_1 = (kh_0)^2\left[\frac{4\epsilon a_2\phi'}{3} + \left(\frac{2}{3}\chi - a_0\right)a_0'\phi' - \left(\frac{2}{3}\chi - a_0\right)\phi''\right] + 0((kh_0)^4) \tag{IV.3}$$

The boundary conditions (1*e*), (1*i*), and (1*j*) lead respectively to

$$b_0 = 0 \tag{IV.4}$$

$$p_0 = -\frac{4}{9}\frac{\sigma k^2 h_0^3}{\rho Q^2}\left[1 - \frac{3}{2}(kh_0)^2\phi'^2\right]\phi'' - \frac{4\nu}{3Q}kh_0(a_0' + a_1\phi') + 0((kh_0)^4) \tag{IV.5}$$

$$a_1 = (kh_0)^2\left[\left(\frac{2}{3}\chi - a_0\right)\phi'' + 3a_0'\phi'\right] + 0((kh_0)^4) \tag{IV.6}$$

In Equations (IV.2) to (IV.6) we have used the relations

$$b_1 = -kh_0a_0', \quad b_2 = -\frac{kh_0}{2}a_1' \tag{IV.7}$$

which are consequences of the continuity equation.

Eliminating p_1 and p_0 from Equation (IV.2) by means of Equations (IV.3) and (IV.5), one obtains‡

$$-\left(\frac{2}{3}\chi - a_0\right)a_0' = \frac{4}{9}\frac{\sigma k^2 h_0^3}{\rho Q^2}\left[\phi''' - \frac{3}{2}(kh_0)^2\left(\phi'^2\phi''' + 2\phi'\phi''^2\right)\right] + \frac{4}{9}\frac{gh_0^2}{kQ^2} - (kh_0)^2\left[\left(\frac{2}{3}\chi - a_0\right)a_0'\phi'^2 - \left(\frac{2}{3}\chi - a_0\right)^2\phi'\phi''\right] + \frac{4}{3}\epsilon\left[a_2 + (kh_0)^2\left(\frac{3}{2}a_0'' - a_2\phi'^2\right)\right] + 0((kh_0)^4) \tag{IV.8}$$

‡ We note that since $h_0^3 = \frac{3\nu Q}{g}$ and $k^2 \approx \frac{\rho g Q}{\nu\sigma}$, there results the factor $\frac{\sigma}{\rho}\frac{h_0^3 k^2}{Q^2} \approx 3$, and therefore it has not the order of $(kh_0)^2$.

Instead of the more exact Equation (IV.8), we have used the third equation from the system (13*a*) in which the terms in $(kh_0)^2$ have been neglected.

By using the results obtained in this work, we have reached the conclusion that the terms left out are negligible, compared to the remaining ones, in any terms of the expansion with respect to ϕ_1 if

$$(kh_0)^2 << 1 \tag{IV.9}$$

If we consider $(kh_0)^2 \leqq 0.10$, one obtains

$$kh_0 \leqq 0.30 \tag{IV.10}$$

In similar manners it is possible to show that for $n \neq 0$ the neglected terms are at least of the order of $(kh_0)^2$ and the remaining ones of the order of unity. The inequality (IV.10) expresses the condition of validity of the thin layer approximation used in this work, therefore the conditions under which the system (3) and the boundary conditions (3) are valid.

Another approximation made in this work consists of neglecting the terms from the expansion (9*a*), beginning with a_8. Since the expansions (9) have been assumed convergent, it is to be expected that the main term neglected is a_8 itself, which may be calculated from the last equation of the system (12*b*) for $m = 3$:

$$56\,\epsilon a_8 = -\left[\left(\chi - \frac{3}{2}a_0\right)a_6' + \frac{15}{2}a_0'\,a_6 + \frac{a_2'a_4}{2} - \frac{9a_4'a_2}{10}\right] \tag{IV.11}$$

By expanding a_8 in Fourier series with respect to z, one obtains

$$a_8 = \sum_{-\infty}^{\infty} E_q\,e^{iqz} \tag{IV.12}$$

The coefficients E_q are then expanded in power series with respect to ϕ_1, under the form

$$E_q = \phi_1^{|q|}\sum_{r=0}^{\infty} E_{q,2r}\,\phi_1^{2r} \tag{IV.13}$$

We have considered that the term a_8 may be neglected for those cases in which

$$\left|\frac{E_{q,2r}}{F_{q,2r}}\right| \leqq \frac{1}{10}, \quad (q, 2r) = (1, 0;\ 2, 0;\ 1, 2;\ 3, 0) \tag{IV.14}$$

where

$|E_{q,2r}|^{v^2} = v^2\hat{E}^2_{q,2r} + \tilde{E}^2_{q,2r}$ and $|F_{q,2r}|^{v^2} = v^2\hat{F}^2_{q,2r} + \tilde{F}^2_{q,2r}$ $|F_{q,2r}|$ being the largest term from the terms $|A_{q,2r}|$, $|B_{q,2r}|$, $|C_{q,2r}|$ and $|D_{q,2r}|$.

From the above condition there results

$$\epsilon_0 \geqq 0.10 \quad (\text{or } \psi \leqq 5.40) \tag{IV.15}$$

In the same manner it may be shown that the term a_4 may be neglected only for $\epsilon_0 \geqq 2$ $(\psi \leqq 0.32)$. For this reason the parabolic approximation may be used only for small Reynolds numbers.

In order to compare the two inequalities under which the present calculations are valid we shall take into account that

$$k \approx k_0(1 + 13\,\phi_1^2), \quad h_0 \approx H_0(1 + 2\,\phi_1^2)$$

Condition $\epsilon_0 \geqq 0.10$ leads to

$$kh_0 \leqq \frac{40(1 + 13\,\phi_1^2)(1 + 2\,\phi_1^2)}{Re} \quad \left(Re \equiv \frac{4Q}{\nu}\right) \tag{IV.16}$$

Condition (IV.16) becomes, beginning with a certain value of Reynolds number, more restrictive than (IV.10).